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A new method for combining NLO QCD with shower Monte Carlo algorithms

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Abstract: I show that with simple extensions of the shower algorithms in Monte Carlo programs, one can implement NLO corrections to the hardest emission that overcome the problems of negative weighted events found in previous implementations. Simple variants of the same method can be used for an improved treatment of matrix element corrections in Shower Monte Carlo programs.

Keywords: QCD, NLO Computations, Parton Model, Hadronic Colliders.

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1. Introduction

Fixed order calculations in Perturbative QCD (PQCD) can be used to compute inclusive quantities in strong processes. The precision of these computations is limited by our ability to compute complex Feynman graphs. Yet, even if we could perform computations at arbitrary order, we would not be able to give predictions for exclusive quantities. This known fact follows from the presence of collinear and soft divergences in fixed order calculations with a definite final state. Only by summing over different final states these divergences can cancel, thereby allowing the computation of certain (i.e. the collinear and infrared insensitive) inclusive quantities.

The only frameworks in which exclusive quantities can be computed must involve the resummation of an infinite class of Feynman graphs. Shower Monte Carlo (SMC) programs perform this resummation in the leading logarithmic (LL) approximation. Going beyond the LL approximation is a considerably complex task: the initial hard process would have to be implemented at the NLO order, and shower development would have to be improved to NLL accuracy, in both the collinear and soft structure. There are however simpler directions in which SMC programs could be improved. Namely, one could stick to the LL approximation as far as the shower development is concerned, but improve the treatment of the hard emission, in such a way that inclusive quantities are predicted with a next-to-leading order (NLO) accuracy. Such an improvement (I will call it SMC+NLO from now on) would clearly merge all the best features of the two approaches. One would have exclusive final state generation together with the accuracy of NLO calculations.

In refs. [1]–[3] a method and implementation of an SMC+NLO program has been given. Such method (referred to as MC@NLO) is based upon a careful elaboration of the NLO results, that has to match certain features of the SMC program. The approximate SMC implementation of the NLO corrections must be subtracted to the exact NLO result in order to avoid double counting. The method of refs. [1]–[3] does not require modifications of the existing shower Monte Carlo code. This is a considerable advantage, because Monte Carlo programs are large and complex, and modifying them is a major task. The method has however few drawbacks:

- The approach is specific to a particular shower Monte Carlo implementation, since the form of the NLO result has to be adapted to it. Similarly, one must extract the NLO terms already present in the Monte Carlo in order to subtract them. Variations in the Monte Carlo implementation (like the new showering variables proposed in refs. [4, 5]) would require computing again both the modified NLO expression and the subtraction term.

- The Monte Carlo may not be fully accurate in the treatment of the soft region. In this case the difference between the exact NLO result and its Monte Carlo approximation may have left over singularities that need special treatment.

- The correction coming from the difference between exact NLO and its MC approximation may be negative. Thus, negative weighted events may be generated.

In the present work, I propose an SMC+NLO method that improves over the method proposed in refs. [1]–[3] on the aspects listed above. This method cannot be directly implemented using existing SMC programs, since it requires minor modifications to the shower development. On the other hand, it is quite simple, and it can also be used to perform matrix element corrections in a more consistent way. It is not unlikely that it may be generalized to implement a SMC+NNLO method, when NNLO calculations will become widely available. It would be desirable that new SMC implementations would eventually provide facilities to implement the method proposed here.

The strategy followed in the present approach is the following. If the hardest (i.e. the largest p_{T}) emission was generated first in the SMC, one would just have to correct the first emission at NLO in order to get an SMC+NLO implementation. Unfortunately, in general this is not the case. A consistent treatment of soft emissions is essential in SMC programs in order to obtain the correct results for hadron multiplicity distributions. Large angle soft radiation from bunches of collinear partons interfere destructively, and the resulting coherent emission is described in SMC programs as arising from the parent parton of the collinear bunch. The corresponding angular ordered shower algorithms can therefore generate soft emissions earlier in the shower [6], so that very often the hardest emission is not the first. The basic result of this work is the construction of a shower algorithm which is fully equivalent to the coherent angular ordered shower, but has the hardest emission generated first. This construction for timelike showers is described in sec. 4. There I show that the hardest emission should be generated with a modified Sudakov form factor, and subsequent emissions should be generated according to the standard algorithm modified by a p_T veto, so that they cannot be harder than the (first generated) hardest emission. Large angle, soft coherent radiation from the particles arising from the hardest splitting must be added, which will result in the presence of truncated showers associated to that pair of particles.

The NLO accurate hardest emission can be easily obtained, as I show in section 5. One simply constructs an NLO Sudakov form factor for the hardest emission in analogy with the corresponding expression in the SMC.

Space-like initial state showers are discussed in detail in section 6. They are usually implemented in the backward evolution formalism [7], i.e. evolving from the hard event back to the incoming hadron. The shower order is from hard to soft, as in the timelike case. This fact allows a similar treatment of the spacelike and timelike cases. Certain specific problems in the spacelike case, having to do with the choice of the parton density scheme, are discussed in detail and shown to be harmless.

The paper is organized as follows. In section 2 I collect the main formulae for the angular ordered shower. In section 3 I introduce a formal definition of the Monte Carlo shower, that will be used in section 4 to construct the equivalent formulation of the shower in which the largest p_{T} emission is generated first. In section 5 the NLO implementation of the hardest emission is discussed. In section 6 the spacelike showers will be discussed. In section 7, the comparison with other matrix element corrections methods will be discussed. In section 8 I give my conclusions and prospects.

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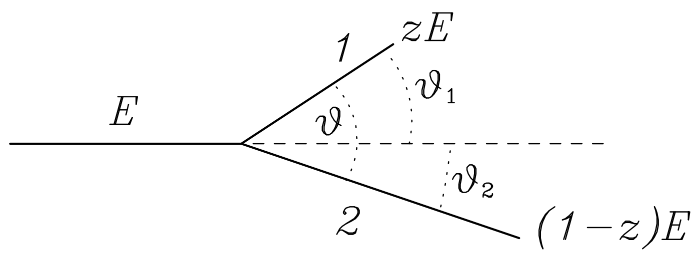


Figure 1: Kinematic variables for a splitting process. E is the energy of the incoming parton; zE and $(1-z)E$ are the energy of parton 1 and 2 respectively.

2. The angular ordered shower

2.1 Kinematics

Shower Monte Carlo algorithms are an effective way of summing up a large class of Feynman graphs in hard QCD collision, including collinear singularities at the leading logarithmic level, and also soft singularities in the double logarithmic region. The implementation given in the HERWIG Monte Carlo program is through an angular ordered shower [6, 8]. We summarize here its general features. We consider the splitting process shown in figure 1. In the limit of small emission angles we have

$$p_T = \theta_1 zE = \theta_2(1-z)E\,, \qquad \theta = \frac{\theta_1}{1-z} = \frac{\theta_2}{z}\,. \tag{2.1}$$

Defining $t = E^2\theta^2$ we then have

$$p_T = \sqrt{t}z(1-z)\,. \tag{2.2}$$

The probability for such a split is given by

$$dP = F(z,t)dz\,dt = \frac{\alpha_S(p_T)}{2\pi}\hat{P}_{ij}(z)\frac{dt}{t}dz\,, \tag{2.3}$$

where $\hat{P}$ is the unregularized Altarelli-Parisi splitting function. The variable z is limited by the requirement $p_T = \sqrt{t}\,z(1-z) > \sqrt{t_0}/4$, and t_0 is the minimum t for a branching to take place. In the angular ordered shower, subsequent branching can take place only with an angle θ less than the previous branching angle. In terms of t, this implies that the maximum t value for a subsequent splitting in figure 1 must be less than z^2t along line 1, and $(1-z)^2t$ along line 2.

2.2 Coherence

In the angular ordered shower it is quite possible to have a large p_T emission following smaller p_T emissions. In the sequence of two branchings depicted in figure 2, for example, one can have z very near one in the first branching, and $z' = 1/2$ in the second branching, so that there is a region of $\theta' < \theta$ for which the second p_T (according to formula (2.2)) is much larger than the first one. The line $(1-z)E$ would be soft in this case, and (if line 2 is a gluon) the splitting kernel eq. (2.3) is singular in this region. Thus it would appear

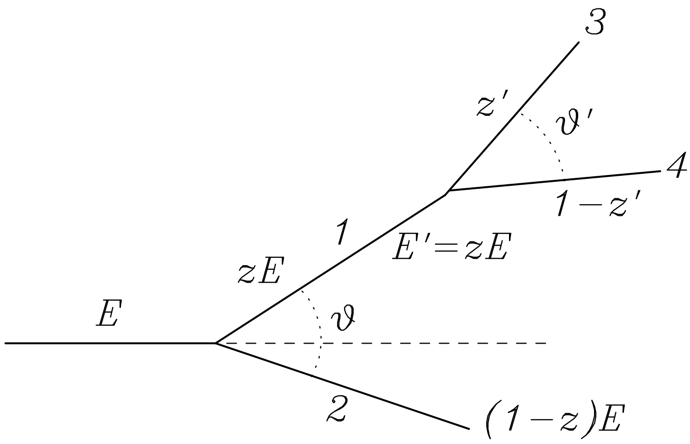


Figure 2: Two subsequent branchings.

that a highly off-shell line can have soft gluon emission singularity. In fact this is not what happens. The soft gluon emission in this region is the result of coherent radiation from lines 3 and 4, and these line have in fact small virtualities.

3. The shower

3.1 Notation

We now want to set up some notation for discussing the properties of a shower. We call $\mathbb{S}(t)$ the angular ordered shower originating from the initial value t. In order to give a more precise meaning to this notion we introduce the basic configuration states[1]

$$|k_1, m_1; \ldots; k_l, m_l\rangle\,, \tag{3.1}$$

where k_i and m_i are the momenta and quantum numbers of the particles, with the normalization[2]

$$\langle k_1, m_1; \ldots; k_l, m_l | k'_1, m'_1; ...; k'_{l'}, m'_{l'}\rangle = \delta_{l,l'} \prod_{i=1}^{l} \delta^3(k_i - k'_i)\delta_{m_i,m'_i}\,. \tag{3.2}$$

A shower is defined as

$$\mathbb{S} = \sum_{l=1}^{\infty} \sum_{m_1 \ldots m_l} \int d^3k_1 \ldots d^3k_l \; C(k_1, m_1; \ldots; k_l, m_l)\; \langle k_1, m_1; \ldots; k_l, m_l| \tag{3.3}$$

where $C(k_1, m_1; \ldots; k_l, m_l)$ is the probability density to generate the state $|k_1, m_1; \ldots; k_l, m_l\rangle$ in the shower. Thus, given an infinitesimal cell around a basic state

$$d\Psi = |k'_1, m'_1; \ldots; k'_{l'}, m'_{l'}\rangle \; d^3k'_1 \ldots d^3k'_{l'} \tag{3.4}$$

[1]We stress that these states do not represent quantum states.

[2]We do not need to symmetrize the states over identical particles. Physical observables will always be symmetric.

the product

$$S \cdot d\Psi \tag{3.5}$$

is the probability to generate a state in the cell $d\Psi$. A final state observable $g(k_1, m_1; \ldots; k_l, m_l)$ will be represented in this notation by

$$\mathbb{G} = \sum_{l=1}^{\infty} \sum_{m_1 \ldots m_l} \int d^3k_1 \ldots d^3k_l \; g(k_1, m_1; \ldots; k_l, m_l) \; |k_1, m_1; k_2, m_2; \ldots; k_l, m_l\rangle \tag{3.6}$$

and the shower is a linear functional over the observable, which yields its average value $\mathbb{S} \cdot \mathbb{G}$.

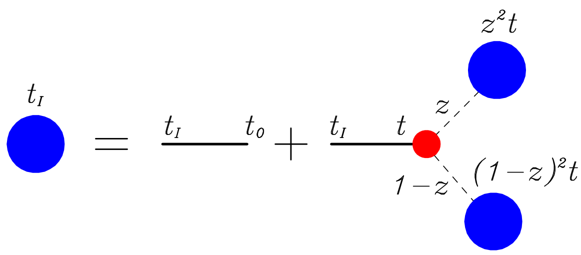
3.2 The shower equation

The angular ordered shower obeys the equation

$$\mathbb{S}(t_I) = \Delta(t_I, t_0)\, \langle \mathbb{I}| + \int_{t_0}^{t_I} \Delta(t_I, t)\, F(z,t)\, \mathbb{S}(z^2 t)\, \mathbb{S}(\,(1-z)^2\, t\,)\, dt\, dz\,, \tag{3.7}$$

where $\langle \mathbb{I}|$ stands for the initial parton $\langle k, m|$, and $\Delta(t_1, t_2)$ is the probability that no emission takes place with the ordering variable between t_1 and t_2. The first term represents the "no emission" case. When we write the product of two showers $\mathbb{S}(z^2 t)\, \mathbb{S}(\,(1-z)^2\, t\,)$ we just mean the obvious thing, i.e. the formal product where the product of basic states is interpreted as a tensor product. We observe that eq. (3.7) is fully meaningful only if, when solved by iteration, it is possible to reconstruct its final state. Thus, we implicitly assume that the parton energy and direction is specified in the shower, and furthermore, that the azimuthal angle in the splitting also appears in the integration. We do not write explicitly these dependences in order to have a lighter notation.

Equation (3.7) can be written graphically as



(3.8)

where the large blobs stands for the shower, thick lines are the Sudakov form factors, and the small blob is the splitting function. The dashed lines represent the connections of the shower blobs to the vertices, and no factors are attached to them. We have omitted to write indices along the lines and vertices of the shower, specifying particle species. They are inessential in the following arguments, and therefore we carry out our discussion assuming that we re dealing with gluons only.

3.3 Unitarity

Shower Monte Carlos guarantee that given the initial t_I one and only one configuration is reached. Thus the observable $\mathbb{G}$ with $g(k_1, m_1; \ldots; k_l, m_l) = 1$ for all l, k, and m must yield $\mathbb{S} \cdot \mathbb{G} = 1$. In other words, unitarity must be satisfied: the sum of the probabilities of all shower configurations must yield one. Thus, from eq. (3.7) we immediately obtain

$$1 = \Delta(t_I, t_0) + \int_{t_0}^{t_I} \Delta(t_I, t)\, F(z, t)\, dt\, dz\,, \tag{3.9}$$

and with the ansatz

$$\Delta(t_1, t_2) = \frac{\Delta(t_1, t_0)}{\Delta(t_2, t_0)}\,, \tag{3.10}$$

we get the solution

$$\Delta(t, t_0) = e^{-\int_{t_0}^{t} F(z,t')\, dz\, dt'}\,, \tag{3.11}$$

which is the usual expression for the Sudakov form factor. Notice that unitarity works only if the expression $\Delta(t_I, t)\, F(z, t) dt dz$ is an exact differential. This is equivalent to saying that $\Delta(t_I, t)$ is the probability that no emission has taken place between t_I and t. Under this condition, the usual probabilistic algorithm of shower Monte Carlo can be used to generate events, i.e. one picks a random number r between 0 and 1, solves the equation $\Delta(t_I, t) = r$ for t, and then generates z with a distribution proportional to $F(z, t)$.

3.4 Vetoed showers

In order to become familiar with the notation we will now prove that the standard vetoing procedure[3] is equivalent to the use of a modified Sudakov form factor. We will use this well known result in the following sections, and thus include its proof for completeness.

A vetoed shower is obtained by applying a constraint $\theta(g(z, t))$ to the branching process. One generates a branching at a scale t, and, if $g(z, t)$ is negative, the branching is rejected, and one generates a new branching starting from t. The equation for a vetoed shower is thus

$$\begin{aligned}\mathbb{S}_V(t_I) \;=\; & \Delta(t_I, t_0)\, \langle \mathbb{I}| + \int_{t_0}^{t_I} \Delta(t_I, t)\, F(z, t)\, \theta(g(z, t))\, \mathbb{S}_V(z^2 t)\, \mathbb{S}_V(\,(1-z)^2\, t\,)\, dt\, dz\, + \\ & + \int_{t_0}^{t_I} \Delta(t_I, t)\, F(z, t)\; [1 - \theta(g(z, t))]\; \mathbb{S}_V(t)\, dt\, dz\,. \end{aligned}\tag{3.12}$$

Using the identity

$$\Delta(t_I, t) = \frac{\Delta(t_I, t_0)}{\Delta(t, t_0)}\,, \tag{3.13}$$

dividing both sides by $\Delta(t_I, t_0)$ and taking a derivative with respect to t_I we get

$$\begin{aligned}\frac{\partial}{\partial t_I} \frac{\mathbb{S}_V(t_I)}{\Delta(t_I, t_0)} \;=\; & \frac{1}{\Delta(t_I, t_0)} \int F(z, t_I)\, \theta(g(z, t_I))\, \mathbb{S}_V(z^2 t_I)\, \mathbb{S}_V(\,(1-z)^2\, t_I\,)\, dz\, + \\ & + \frac{\mathbb{S}_V(t_I)}{\Delta(t_I, t_0)} \int F(z, t_I)\; [1 - \theta(g(z, t_I))]\; dz\,. \end{aligned}\tag{3.14}$$

[3] See for example ref. [9].

Defining now

$$r(t_I, t_0) = \exp\left(-\int_{t_0}^{t_I} F(z,t)\ [1 - \theta(g(z,t))]\ dt\, dz\right), \tag{3.15}$$

we have

$$\frac{\partial r(t_I, t_0)}{\partial t_I} = -r(t_I, t_0) \int F(z, t_I)\ [1 - \theta(g(z, t_I))]\ dz\,, \tag{3.16}$$

so that eq. (3.14) becomes equivalent to

$$\frac{\partial}{\partial t_I} \frac{r(t_I, t_0)\, \mathbb{S}_V(t_I)}{\Delta(t_I, t_0)} = \frac{r(t_I, t_0)}{\Delta(t_I, t_0)} \int F(z, t_I)\, \theta(g(z, t_I))\, \mathbb{S}_V(z^2 t_I)\, \mathbb{S}_V(\,(1-z)^2\, t_I\,)\, dz\,. \tag{3.17}$$

Defining now

$$\Delta'(t_I, t_0) = \frac{\Delta(t_I, t_0)}{r(t_I, t_0)} = \exp\left(-\int_{t_0}^{t_I} F(z,t)\, \theta(g(z,t))\, dt\, dz\right), \tag{3.18}$$

and integrating eq. (3.17) using the initial condition of eq. (3.12) we get

$$\mathbb{S}_V(t_I) = \Delta'(t_I, t_0)\, \langle \mathbb{I}| + \int_{t_0}^{t_I} \Delta'(t_I, t)\, F(z,t)\, \theta(g(z,t))\, \mathbb{S}_V(z^2 t)\, \mathbb{S}_V(\,(1-z)^2\, t\,)\, dt\, dz \tag{3.19}$$

which is a shower equation with a modified Sudakov form factor which matches precisely the splitting vertex.

In the following, we will encounter showers with Sudakov form factors and splitting vertices containing a theta function that limits the transverse momentum. These showers can be simply implemented by vetoing, without the need to modify the form factor calculation procedure in a SMC.

4. The largest p_{T} emission

We would like now to transform the angular ordered equation in such a way that the hardest emission is generated first. The first step to perform such transformation is to prove the following three statements:

(I) *the largest p_{T} emission in an angular ordered shower always takes place along the hardest line in the shower.* (The hardest line in the shower is the one that starts at the beginning of the shower and always follows the largest z line in the splitting vertices.)

(II) *along the hardest line, configurations with non-soft emission before the hardest emission are collinear subleading.*

(III) *along the hardest line, for leading configurations, from the first non-soft emission down to the last we have $t \lesssim p_{\mathrm{T}}^2$ (i.e. t of order p_{T}^2 or smaller).*

The proof of (I) is quite simple. Let us call t, z the emission variables of the largest p_{T} splitting. There is a unique way to walk backward in the shower, up to the initial t_I. Let us call $t_1, z_1, \ldots t_l, z_l$ all the splitting occurring along this line before t, z, according

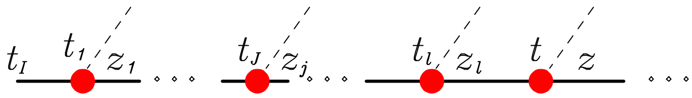


Figure 3: Hardest $p_{\rm T}$ occurring at t, z, in a shower initiated at t_I.

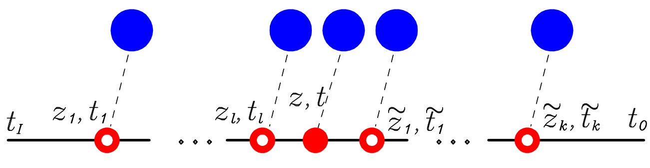
to figure 3. Consider any splitting t_j, z_j before t, z. We must have $t < z_j^2 t_j$. If $p_{\rm T} > p_{{\rm T}j}$ then $tz^2(1-z)^2 > t_j z_j^2(1-z_j)^2$, which implies $(t_j z_j^2)z^2(1-z)^2 > t_j z_j^2(1-z_j)^2$, and thus $z^2(1-z)^2 > (1-z_j)^2$, which immediately implies $1 - z_j < 1/4$, and thus $z_j > 3/4$ for all j, which proves our lemma.

The proof of (II) goes as follows. Suppose there is a non-soft emission at t_k, z_k before the hardest emission. Non-soft means $1 - z_k \approx 1$. Also $z_k > 1/2$, therefore from $\sqrt{t_k} z_k(1-z_k) < p_{\rm T}$ we get $t_k \lesssim p_{\rm T}^2$ and *a fortiori* $t_k \lesssim t$. But by angular ordering we must have $t_k > t$. Thus none of the collinear integrals from t_k up to t has a large logarithmic range, and so these configurations are subleading.

The proof of (III) goes as follows. The first non-soft emission must be the hardest emission itself, or may occur after it, but not earlier because of statement (II). Let us call t', z' the values corresponding to the first non-soft emission. We must have $\sqrt{t'}z'(1-z') < p_{\rm T}$, which implies $t' \lesssim p_{\rm T}^2$. By angular ordering the same must hold for all following emissions.

The equation for the angular ordered shower (3.7) can be formally solved by iteration. Using the fact that the largest $p_{\rm T}$ emission has to be on the hardest line, we can write a formal solution where the largest $p_{\rm T}$ is explicitly present:

$$\mathbb{S}(t_I) = \Delta(t_I, t_0)\langle \mathbb{I}| + \sum_{l,k=0}^{\infty} \int$$



. (4.1)

The z, t vertex has the largest $p_{\rm T}$. In the following we will also denote collectively the whole z, t variable sequence as z_i, t_i with $i = 1, \ldots, l+k+1$, where $z, t = z_{l+1}, t_{l+1}$, and $z_{l+1+j}, t_{l+1+j} = \tilde{z}_j, \tilde{t}_j$.

The graphic expression in figure 4.1 has the following meaning:

- Thick lines are Sudakov form factors; more specifically, they are given by

$$\Delta(t_I, t_1),\ \Delta(z_1^2 t_1, t_2), \ldots, \Delta(z_{l+k+1}^2 t_{l+k+1}, t_0)\,. \tag{4.2}$$

- The large solid blobs are $\mathbb{S}((1-z_i)^2 t_i^2)$,
- The hollow vertex blobs stand for

$$2F(z_i, t_i)\,\theta(z_i - 1/2)\,\theta(p_{\rm T} - \sqrt{t_i} z_i(1-z_i))\,. \tag{4.3}$$

where we have defined $p_{\rm T} = \sqrt{t}z(1-z)$.

- The solid vertex blob (i.e. the largest p_{T} vertex) is given by the same expression as the hollow blob without the theta function.

- All intermediate z_i, t_i are integrated.

The factor of 2 accounts for the fact that the largest p_{T} can occur on either line of the splitting. Larger p_{T} emissions cannot occur along the $1-z_i$ (and $1-\tilde{z}_i$) lines, according to our statement (I).

Equation (4.1) cannot be implemented as it stands in a Monte Carlo algorithm: the Sudakov form factors do not match with the splitting vertices, because of the p_{T} theta function. We thus perform the following manipulation. We rewrite the Sudakov form factors as

$$\begin{aligned}\Delta(t_i z_i^2, t_{i+1}) &= e^{-\int_{t_{i+1}}^{t_i z_i^2} dt' \int dz' F(z',t')} \\ &= e^{-\int_{t_{i+1}}^{t_i z_i^2} dt' \int_{1/2}^{1} dz' 2F(z',t')\,\theta(p_{\mathrm{T}} - z'(1-z')\sqrt{t'})} \times \\ &\quad \times e^{-\int_{t_{i+1}}^{t_i z_i^2} dt' \int_{1/2}^{1} dz' 2F(z',t')\theta(z'(1-z')\sqrt{t'} - p_{\mathrm{T}})}\,. \end{aligned} \tag{4.4}$$

The first factor on the right hand side of eq. (4.4) matches precisely the splitting vertex in eq. (4.1). The second factor is a remnant. Because of statements (II) and (III) the remnant can always be written as

$$e^{-\int_{t_{i+1}}^{t_i z_i^2} dt' \int_{1/2}^{1} dz' 2F(z',t')\theta(z'(1-z')\sqrt{t'} - p_{\mathrm{T}})} \approx e^{-\int_{t_{i+1}}^{t_i} dt' \int dz' F(t',z')\theta(z'(1-z')\sqrt{t'} - p_{\mathrm{T}})}\,. \tag{4.5}$$

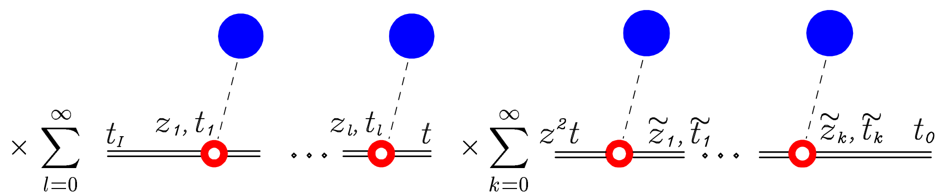
In fact, assume that k is the first non-soft vertex. Because of statement (II) in all previous vertices we can replace $z_i \to 1$, and eq. (4.5) holds. According to statement (III), starting with the first non-soft vertex the t values are not larger than p_{T}, so that the exponent in eq. (4.5) vanishes because of the theta function, and also in this case eq. (4.5) is trivially satisfied.

We will call $\Delta_R(t_I, p_{\mathrm{T}})$ the product of all remnants. From eq. (4.5) we obtain

$$\Delta_R(t_I, p_{\mathrm{T}}) \equiv e^{-\int_{t_0}^{t_I} dt' \int dz' F(z',t')\theta(z'(1-z')\sqrt{t'} - p_{\mathrm{T}})}\,. \tag{4.6}$$

Equation (4.6) has an obvious physical interpretation. It is the Sudakov form factor for not emitting a particle with transverse momentum larger than p_{T}. It can be used to generate the largest p_{T} emission first, in association with the $F(z,t)\,dz\,dt$ factor present in eq. (4.1). We can thus rewrite eq. (4.1) in the following form

$$\mathbb{S}(t_I) = \Delta(t_I, t_0)\langle \mathbb{I}| + \int_{t_0}^{t_I} dt\,\theta(z > 1/2)\,dz\,\Delta_R(t_I, p_{\mathrm{T}}) 2F(z,t)\;\mathbb{S}((1-z)^2 t)$$



. (4.7)

Double lines are the modified Sudakov form factors, defined as

$$\Delta_V(t_i z_i^2, t_{i+1}) = e^{-\int_{t_{i+1}}^{t_i z_i^2} dt' \int_{1/2}^{1} dz' 2F(z',t')\,\theta(p_{\rm T} - z'(1-z')\sqrt{t'})} \,. \tag{4.8}$$

The two graphic factors in eq. (4.7) are now showers, that can be easily implemented in a Monte Carlo algorithm. In fact they are angular ordered showers with the addition of a $p_{\rm T}$ veto. The veto needs to be applied only along the hardest line, but since no emissions with transverse momenta larger than $p_{\rm T}$ are possible in any other branches of the showers, we might as well apply the veto everywhere, without the need to follow the hardest line.

Our final result for the shower is

$$\begin{aligned} \mathbb{S}(t_I) = \Delta(t_I, t_0)\langle \mathbb{I}| + \int_{t_0}^{t_I} dt \int_0^1 dz\, \Delta_R(t_I, p_{\rm T}) F(z,t)\; \mathbb{S}_V((1-z)^2 t, p_{\rm T}) \times \\ \times\, \mathbb{S}_{VT}(t_I, t, p_{\rm T})\; \mathbb{S}_V(z^2 t, p_{\rm T})\,, \end{aligned} \tag{4.9}$$

where

- $\mathbb{S}_V((1-z)^2 t, p_{\rm T})$ and $\mathbb{S}_V(z^2 t, p_{\rm T})$ are angular ordered $p_{\rm T}$ vetoed showers.
- $\mathbb{S}_{VT}(t_I, t, p_{\rm T})$ is also an angular ordered $p_{\rm T}$ vetoed shower, starting at t_I, except that along the hardest line the minimal scale for showering is t instead of t_0. We call it the truncated shower.

Unitarity of eq. (4.9) works as follows. Since both $\mathbb{S}_{VT}(t_I, t, p_{\rm T})$ and $\mathbb{S}_V(t, t_0)$ project to 1 by unitarity, the unitarity equation is

$$1 = \Delta(t_I, t_0) + \int_{t_0}^{t_I} dt \int_0^1 dz\, \Delta_R(t_I, p_{\rm T}) F(z,t)\,, \tag{4.10}$$

which is the analogue of eq. (3.9). We have

$$\begin{aligned} &\int_{t_0}^{t_I} dt \int_0^1 dz\, \Delta_R(t_I, p_{\rm T}) F(z,t) = \\ &\quad = \int dp_{\rm T} \int_{t_0}^{t_I} dt \int_0^1 dz\, \delta(p_{\rm T} - \sqrt{t} z(1-z))\, F(z,t) e^{-\int_{t_0}^{t_I} dt' \int dz' F(z',t') \theta(z'(1-z')\sqrt{t'} - p_{\rm T})} \\ &\quad = \int dp_{\rm T} \frac{d}{dp_{\rm T}} e^{-\int_{t_0}^{t_I} dt' \int dz' F(z',t') \theta(z'(1-z')\sqrt{t'} - p_{\rm T})} = 1 - \Delta(t_I, t_0)\,. \end{aligned} \tag{4.11}$$

The last step follows from the fact that the value of the exponential is 1 for $p_{\rm T} \to \infty$ (because the exponent vanishes), and it reduces to $\Delta(t_I, t_0)$ for $p_{\rm T} = 0$, since in this case the theta function is irrelevant.

The first emission in eq. (4.9) is given in terms of a Sudakov form factor that depends upon z and t, through the $p_{\rm T}$ variable. It can be easily expressed in terms of $p_{\rm T}$ as an independent variable

$$\int_{t_0}^{t_I} \frac{dt}{t} \int_0^1 dz \Longrightarrow \int_{\sqrt{t_0}/4}^{\sqrt{t_I}/4} \frac{2\, dp_{\rm T}}{p_{\rm T}} \int_0^1 dz\, \theta(\sqrt{t_I} - t)\,, \tag{4.12}$$

where now $t = p_{\rm T}^2 \,/\, z^2(1-z)^2$ is the dependent variable. Observe that the lower bound in $p_{\rm T}$ is $\sqrt{t_0}/4$. As discussed earlier, we always have an implicit cut-off at this value of $p_{\rm T}$ that prevents the argument of $\alpha_{\rm S}$ from becoming too small.

We can express the first emission Sudakov form factor in terms of a $p_{\rm T}$ integral

$$\Delta_R(t_I, p_{\rm T}) = \exp\left[-\int_{p_{\rm T}}^{\sqrt{t_I}/4} dp'_{\rm T} \int dz' \, \frac{2t'}{p'_{\rm T}} F(z',t')\theta(t_I - t')\right]. \tag{4.13}$$

Now $\Delta_R(t_I, p_{\rm T})$ can be used to generate the largest $p_{\rm T}$ emission using the standard SMC technique: one generates a random number $0 < r < 1$ and solves the equation $\Delta_R(t_I, p_{\rm T})$ for $p_{\rm T}$. Alternatively, one can define

$$\tilde{\Delta}_R(p_{\rm T}^{\rm max}, p_{\rm T}) = \exp\left[-\int_{p_{\rm T}}^{p_{\rm T}^{\rm max}} dp'_{\rm T} \int dz' \, \frac{2t'}{p'_{\rm T}} F(z',t')\right], \tag{4.14}$$

generate the event according to $\tilde{\Delta}_R$, and implement the $\theta(t_I - t')$ by vetoing.

The Sudakov form factors in $\mathbb{S}_{VT}(t_I, t, p_{\rm T})$ and $\mathbb{S}_V(t, p_{\rm T})$ include the $p_{\rm T}$ theta function. They depend therefore upon two variables, t and $p_{\rm T}$. They can be easily implemented using the standard Sudakov form factor (i.e. without the theta function), and a $p_{\rm T}$ veto procedure as in eq. (3.12) with $g(z,t) = p_{\rm T} - \sqrt{t}z(1-z)$.

The truncated shower is a soft shower. It does not appreciably degrade the energy entering the hardest emission. This fact is necessary in order to generate the hardest emission first. It is also natural: the radiation of the truncated shower is in reality coherent radiation from final state particles, and so it should not steal energy from the hardest splitting.

Observe that hard radiation in the truncated shower does not have any logarithmic enhancement (i.e. neither soft nor collinear). Formally, this works as follows. If z is not near 1 we can only have collinear logarithms. Because of the angular ordering and the p_T veto we have

$$\theta < \theta_i\,, \quad E\,\theta_i z_i(1-z_i) < (z_i E)\,\theta\, z(1-z) < \frac{z_i\, E\theta}{4} \implies \theta < \theta_i < \frac{\theta}{4(1-z_i)} \tag{4.15}$$

(which forces $z > 3/4$). In order to have a logarithmic integral we need a large θ_i range, and this is possible only if $z \to 1$. Thus no logarithmic enhancement is present for hard radiation.

5. NLO Corrections

5.1 NLO expansion of the SMC

The argument given in the previous Section was referring to a single jet being produced. The extension to the general case is straightforward. If there are m primary partons, one introduces the total Sudakov form factor

$$\Delta_R(\{t_I\}, p_{\rm T}) = \prod_{i=1,m} \Delta_R^i(t_I^i, p_{\rm T})\,, \tag{5.1}$$

where with $\{t_I\}$ we denote the set of the initial showering variables t_I^i, $i = 1 \ldots m$ for all primary partons. The probability distributions for the hardest emission is given by

$$
\begin{aligned}
d\sigma &= B(p_1 \ldots p_m) d\Phi_m \times \\
&\times \left[\Delta_R(\{t_I\}, 0) + \Delta_R(\{t_I\}, p_{\mathrm{T}}) \sum_{i=1,m} F_i(z,t)\, \theta(t_I^i - t)\, dz\, dt\, \frac{d\phi}{2\pi} \right], \qquad (5.2)
\end{aligned}
$$

where $B(p_1 \ldots p_m)$ is the differential cross section for the process at the Born level. The derivation of equation (5.2) is a straightforward extension of the argument presented in section 4. Notice that in each emission term in eq. (5.2), according to eq. (5.1), there are $\Delta_R(t_i, p_{\mathrm{T}})$ factors also for the primary partons that are not emitting. These factors compensates for the p_{T} veto that is applied also to their showers.

The Monte Carlo generation of the event proceeds as follows. One generates the hardest p_{T} according to $\Delta_R(\{t_I\}, p_{\mathrm{T}})$. The i and z values[4] are chosen with a probability proportional to $F_i(z,t)\, \theta(t_I^i - t)$, where $t = p_{\mathrm{T}}^2/z^2/(1-z)^2$. Once i and z are chosen, we construct the two partons originating from i. We let all the $m+1$ partons shower with a p_{T} veto, and add a truncated shower, starting from t_I^i down to t, originating from i.

Equation (5.2) has the following $\mathcal{O}(\alpha_{\mathrm{S}})$ expansion

$$
\begin{aligned}
d\sigma &= B(p_1 \ldots p_m) d\Phi_m \left[1 - \sum_{i=1,m} \int_{t_0}^{t_I^i} F_i(z,t)\, dz\, dt + \sum_{i=1,m} F_i(z,t)\, \theta(t_I^i - t)\, dz\, \frac{d\phi}{2\pi} dt \right] \\
&= B(p_1 \ldots p_m) d\Phi_m \left[1 + \sum_{i=1,m} F_i(z,t)_+\, \theta(t_I^i - t)\, dz\, \frac{d\phi}{2\pi} dt \right], \qquad (5.3)
\end{aligned}
$$

where the first two terms in the square bracket arise from the expansion of Δ_R. The + notation means that the singularities in t and z are regulated according to the + prescription, i.e. in such a way that

$$
\int_{t_0}^{t_I^i} dt dz F_i(z,t)_+ = 0 \,. \qquad (5.4)
$$

The + prescription guarantees that unitarity is preserved also at the perturbative level, through the familiar cancellation of real and virtual singularities. In fact, the second term in the square bracket of eq. (5.3) is the virtual correction. Observe that also in eq. (5.2), by unitarity, the factor in the square bracket integrates to one, so that the cancellation mechanism works to all order in the perturbative expansion.

5.2 Parton event generator

In a "parton" event generator the $\mathcal{O}(\alpha_{\mathrm{S}})$ approximation to the radiation process would be implemented by associating to an event with radiation, characterized by the variables $p_1 \ldots p_n, z, t, \phi$, a counter-event with opposite (negative) weight, and n body kinematics $p_1 \ldots p_n$. This association (or *projection*) is the inverse operation of what the SMC does

[4] We assume for simplicity an isotropic distribution in the azimuth ϕ of the radiated particle.

when splitting one particle into two. In the splitting process the SMC substitutes the $i^{\rm th}$ particle in the ensemble $p_1 \ldots p_m$ with two new particles, with momenta computed as a function of p_i, z, t and ϕ. After this, some momentum reshuffling is needed, since the sum of the four-momenta of the splitting products now has a non vanishing invariant mass, and energy-momentum conservation must be enforced.

The SMC may generate radiation from different primary partons in overlapping regions. In this case a radiated parton may be associated with more than one of the m primary partons. Thus, in general, for a given event with one emission, there can be m projections and m counterterms, one for each association of the radiated parton with one primary parton.

NLO calculations performed using the subtraction method often associate to the event with radiation few counter-events with no radiation, according to some projections that are chosen for convenience.[5]

The $\mathcal{O}(\alpha_{\rm S})$ expansion of the SMC given in term of the subtraction method, as presented here, was first obtained in ref. [1], where it plays a fundamental role in the computation of the term to subtract from the NLO cross section in order to avoid overcounting. In the present work, our aim will be to substitute the SMC NLO result with the exact one, and then work our way backward to the (NLO accurate) analogue of eq. (5.2).

5.3 Exact NLO formula

The exact next-to-leading expression for the cross section can be described schematically by the following formula

$$\begin{aligned} d\sigma &= B(p_1 \ldots p_m) d\Phi_m + V(p_1 \ldots p_m) d\Phi_m + \\ &+ \left[R(p_1 \ldots p_{m+1}) d\Phi_{m+1} - \sum_i C_i(p_1 \ldots p_{m+1}) d\Phi_{m+1} \mathbb{P}_i \right] \end{aligned} \tag{5.5}$$

which should be read as follows: the Born and virtual terms generate m body events, the real emission term generates $m+1$ body events, weighted by R, with some associated counter-events with weights C_i, and m body kinematics obtained applying the projections $\mathbb{P}_i$ to the $m+1$ body event. The projection is required to be IR and collinear insensitive, so that if one parton becomes soft the projected configuration is obtained from the full configuration by removing the soft parton, and if two partons become collinear the projected configuration is obtained by merging the two collinear partons. Furthermore, the difference between R and $\sum C_i$ is non-singular.

5.4 SMC+NLO strategies

In the MC@NLO implementation [1]–[3], the Monte Carlo hardest emission is corrected to match the NLO calculation. This is done by the following procedure:

(i) The NLO cross section formula is rewritten using a projection $\mathbb{P}$ that coincides with the shower Monte Carlo projection.

[5] A specific illustration of this method is given in ref. [10, section 4].

(ii) The shower Monte Carlo approximation to the NLO cross section, eq. (5.3) is subtracted to the exact NLO cross section (5.5). This subtraction is possible since the two formulae, after step (i), have the same projection.

(iii) The Monte Carlo is run with its standard input, plus a correction obtained by adding a contribution where the subtracted NLO correction of step (ii) is used as an input to the Monte Carlo shower.

The advantage of this method is that it does not require to modify the showering code in order to be implemented. Observe that the NLO correction in (ii) is always a hard correction, since collinear and soft singularities have been removed from the NLO formula by subtracting the Monte Carlo NLO approximation. The disadvantages of this approach are

- The NLO calculation has to be tuned to the Monte Carlo, by changing the projection $\mathbb{P}$. It becomes therefore Monte Carlo dependent.
- One has to determine the Monte Carlo NLO approximation, a task that is not always simple.
- The Monte Carlo implementation of the soft limit may be less than perfect. Thus, one has to make sure that the remaining IR sensitivity in the difference between the exact and the Monte Carlo NLO expression does not have sensible consequences. This last problem is in fact, in practice, a very minor problem.
- The NLO correction needs not be positive. Thus, one is forced to accept negative weighted events.

If the hardest emission in a shower Monte Carlo is generated first, one has the possibility of replacing the first emission with the exact one. We now illustrate schematically how this can be done. For ease of illustration, we assume that the Born final state is described by the variables $v_1 \dots v_l$, and the final state with a real emission is described by $v_1 \dots v_l, r_1, r_2, r_3$, where the variables r_i are associated with the radiated parton. We assume that the projection is simply $\mathbb{P}\{v_1 \dots v_l, r_1, r_2, r_3\} \to \{v_1 \dots v_l\}$. The phase space is written as $d\Phi_{m+1} = d\Phi_v \, d\Phi_r$ where $d\Phi_v$ is the Born phase space, and $d\Phi_r$ is Πdr_i times a suitable Jacobian. We now write the NLO exact formula in the following way

$$\begin{aligned} d\sigma &= B(v) d\Phi_v + V(v) d\Phi_v + [R(v,r) d\Phi_v d\Phi_r - C(v,r) d\Phi_v d\Phi_r \mathbb{P}] \\ &= [V(v) + (R(v,r) - C(v,r))\, d\Phi_r \mathbb{P}]\, d\Phi_v + B(v) d\Phi_v \left[1 + \frac{R(v,r)}{B(v)} (1-\mathbb{P})\, d\Phi_r\right]. \end{aligned} \quad (5.6)$$

Comparing eqs. (5.2) and (5.3), we immediately see that the analogue of eq. (5.2) arising from eq. (5.6) is given by

$$\begin{aligned} d\sigma = &\, [V(v) + (R(v,r) - C(v,r))\, d\Phi_r \mathbb{P}]\, d\Phi_v + \\ &+ B(v) d\Phi_v \left[\Delta_R^{(\mathrm{NLO})}(0) + \Delta_R^{(\mathrm{NLO})}(p_{\mathrm{T}}) \frac{R(v,r)}{B(v)} d\Phi_r\right], \end{aligned} \quad (5.7)$$

where we have defined

$$\Delta_R^{(\mathrm{NLO})}(p_\mathrm{T}) = e^{-\int d\Phi_r \frac{R(v,r)}{B(v)} \theta(k_\mathrm{T}(v,r)-p_\mathrm{T})} . \tag{5.8}$$

One can implement eq. (5.7) in an SMC+NLO implementation by generating Born events with distribution $B(v_1 \ldots v_l)$, generating the first emission according to the second line of eq. (5.7), and then generating the subsequent emissions as p_T vetoed shower. Furthermore, one should associate a truncated vetoed shower from the combined emitted parton and the closest (in p_T) primary parton. The first term in eq. (5.7) can be generated independently, and attached to an ordinary shower, since it is formally of higher order in α_S. With this method, negative weighted events could be generated, since this term is not guaranteed to be positive. A better procedure would be the following. One defines

$$\bar{B}(v) = B(v) + V(v) + \int (R(v,r) - C(v,r))\, d\Phi_r \tag{5.9}$$

and then implements the hardest emission as

$$d\sigma = \bar{B}(v) d\Phi_v \left[\Delta_R^{(\mathrm{NLO})}(0) + \Delta_R^{(\mathrm{NLO})}(p_\mathrm{T}) \frac{R(v,r)}{B(v)} d\Phi_r \right] . \tag{5.10}$$

Eq. (5.10) overcomes the problem of the negative weights, in the sense that the region where $\bar{B}$ is negative must signal the failure of perturbation theory, since the NLO negative terms have overcome the Born term.

The structure of the counterterm and the projection in NLO calculations is in general more involved than in the example illustrated above. However, one can separate the real contribution into several term, each one of them singular in a particular collinear region.[6] To each term one can associate a counterterm with a projection, and choose v and r variables in such a way that the projection leaves the v variables unchanged, and the phase space has the factorized form $d\Phi_v d\Phi_r$. The discussion given above would go through unchanged, except that in the exponent of (5.8) and in (5.10) several terms would appear, one for each singular region, instead of the single R/B term, in closer analogy with eqs. (5.1) and (5.2).

5.5 Alternative formulations

Several variants of eq. (5.10) are possible. First of all, one may replace B with $\bar{B}$ in the square bracket and in the Sudakov form factor, the difference being of NNLO order.

It is not strictly necessary to have the exact expression for R in eqs. (5.8) and (5.7). We can choose an $\tilde{R} \leq R$, and add back the positive difference $R - \tilde{R}$ with standard Monte Carlo methods, provided that the difference is non-singular, i.e. that $\tilde{R}$ has the same singularity structure of R. For example, we may want $\tilde{R}$ to smoothly vanish outside of the singular regions, in order to avoid the unnecessary exponentiation of large NLO corrections

[6] For example, defining $R_k = \frac{1}{\sum_i \frac{1}{S_i}} \frac{1}{S_k}$, where S_k is the mass of the pair formed by the k^{th} parton with the radiated parton, we have $\sum R_i = R$, and each R_k is singular only in the region where the emitted parton is collinear to the k^{th} parton, or soft.

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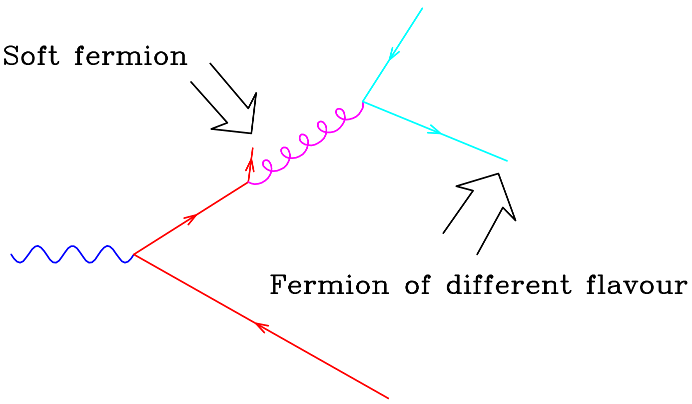


Figure 4: If one follows the hardest line instead of the fermion line, a hardest emission which is not present at the NLO level may be generated.

unrelated to the singular region. This is easily obtained by defining $\tilde{R} = R \times h(p_{\mathrm{T}})$ where $h(p_{\mathrm{T}})$ goes to one as $p_{\mathrm{T}} \to 0$ and vanishes more or less rapidly when p_{T} is away from zero. There will be a left over contribution $R - \tilde{R}$, which is positive and non-singular, and can be added separately.

5.6 Running coupling

In standard NLO calculations one usually uses a coupling constant evaluated at a fixed renormalization scale, of the order of some characteristic hard scale in the process. The SMC's use instead a running scale for the emission vertex, of the order of p_{T}. In order for the NLO corrected hardest emission not to spoil the all order features of the SMC, it is convenient therefore to perform an analogous choice in eq. (5.10). This can be easily done by taking equal to p_{T} the argument for one power of α_{S} (i.e. the one associated with the next-to-leading order radiation) in the expression for R wherever it appears inside the square bracket of eq (5.10) (i.e. also in $\Delta_R^{(\mathrm{NLO})}$). By doing this one gets the correct LL Sudakov suppression of events with a small largest p_{T}. It is easy to see that this does not spoil the NLO accuracy of the result, since it affects a term which is already of next-to-leading order.

5.7 Emission from fermions

Following the hardest line in order to find the hardest emission is only necessary for gluon emission lines. Since all emissions before the hardest must be soft, if the soft particle emitted is not a gluon one has power suppression of the corresponding configuration, which can therefore be discarded. Thus, in the case of a fermion line, one can simply follow the fermion line. In fact, by not doing so one generates configurations that are not even present in the NLO corrections. The example of figure 4 clarifies this issue. If one follows the hardest line instead of the fermion line one can have a hardest emission of a fermion of different flavour, which is not present at the NLO level. For the same reason, if one reaches a $g \to q\bar{q}$ splitting while following a gluon line, one should not proceed any further. In fact, if the hardest emission arises after this, according to our statement (II) the $1 - z$

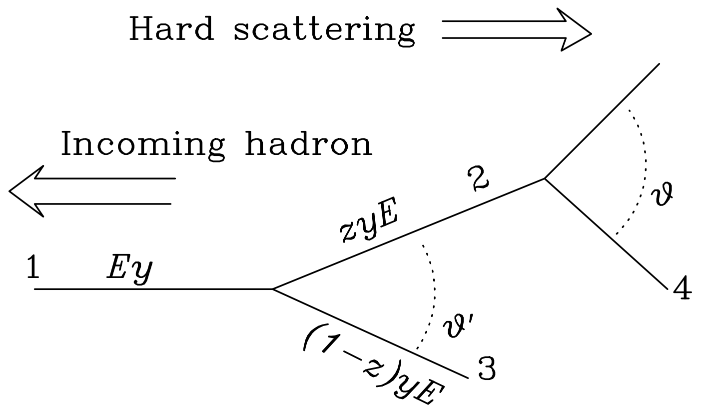


Figure 5: Kinematic variables for a spacelike splitting process. As for the timelike case, Ey, zyE and $(1-z)yE$ are the energies of partons 1,2 and 3.

value of the gluon splitting is forced to be small, thus yielding a suppressed configuration, since the gluon splitting process is not soft-singular.

6. Initial state showers

6.1 Kinematics

We will now study the Monte Carlo equation for an initial state shower. We denote with $\bar{\mathbb{S}}_i(t,x)$ the backward shower from a parton of type i, momentum fraction x and scale variable t. The kinematics of a branching is represented in figure 5. The emissions are ordered in increasing angles towards the hard scattering. Also in the case of initial state showers the angular ordering implements interference effects in soft gluon emission. In figure 5, for example, in the region where $\theta \gg \theta'$ and for soft (gluon) emission, particle 4 is emitted coherently from particles 1 and 3, and since the charge difference of 1 and 3 is the charge of 2 it appears as if it was emitted from 2.

We consider the region where x (and therefore also all z's) is not small, that is to say we ignore the small-x problem now. The evolution variable can then be taken to be

$$t = E^2\theta^2\,, \tag{6.1}$$

so that, with the energy assignements of figure 5 and $t' = E^2\theta'^2$, the transverse momentum in the splitting is given by

$$p_{\mathrm{T}} = \sqrt{t'}yz(1-z)\,. \tag{6.2}$$

6.2 Backward evolution

In the backward evolution formalism [7] the probability of evolving backward from t, x to t', x without emitting is given by [8]

$$\Pi(t,t',x) = \frac{f(x,t')}{f(x,t)}\,\Delta(t,t') = \frac{f(x,t')\,\Delta(t)}{f(x,t)\,\Delta(t')}\,, \tag{6.3}$$

where f is the parton density. It is thus the same as in the timelike case, except for the factor $f(x,t')/f(x,t)$, which accounts for the different probability to find a parton with a fraction x of the incoming momentum at the scale t' instead of the scale t. As $t' \to t_0$ the probability $\Pi(t,t',x)$ goes to its minimum, and for $t' \to t$ it goes to 1. In order for the formalism to work Π must also be a monotonic function of t'. It is interesting to see how this works, by taking the derivative of $\Pi(t,t',x)$ with respect to t'

$$t'\frac{\partial}{\partial t'}\Pi(t,t',x) = \frac{\Pi(t,t',x)}{f(x,t')}\int dz\,\left[t'F(z,t')f(x,t') + \frac{\alpha_S}{2\pi}P(x/z)f(z,t')\frac{1}{z}\theta(z-x)\right], \quad (6.4)$$

where we have used the Altarelli-Parisi equation for f. Since

$$F(z,t') = \frac{\alpha_S}{2\pi t'}\hat{P}(z)$$

we see that the first term on the right hand side of eq. 6.4 is exactly what is needed to turn the regularized splitting kernel $P(x/z)$ in the second term into the unregularized one (thus leaving a positive definite expression in the square bracket) provided the argument of α_S in the evolution equation matches the one used in $F(z,t)$. Notice that we have also assumed that the evolution variable of the Altarelli Parisi equation is the same variable used in the shower evolution. With these assumptions we thus have

$$\frac{\partial}{\partial t'}\Pi(t,t',x) = \Pi(t,t',x)\frac{1}{f(x,t')}\int_x^1 \frac{dz}{z}F(x/z,t')f(z,t')\,. \quad (6.5)$$

From eq. (6.5) we derive the alternative form of the no-branching probability in backward evolution

$$\Pi(t,t',x) = \exp\left\{-\int_{t'}^{t} dt''\int\frac{dz}{z}\,F(x/z,t'')\frac{f(z,t'')}{f(x,t'')}\right\}, \quad (6.6)$$

which is the form proposed in the paper where backward evolution was first introduced [7].

The spacelike shower equation for backward evolution is given by

$$\begin{aligned}\bar{\mathbb{S}}_i(t,x) \;=\; \Pi_i(t,t_0,x) + \int_{t_0}^{t} dt'\,dz\,dy\,\Pi_i(t,t',x)\,F_{ij}(t',z)\delta(x-zy)\,\times \\ \times\;\bar{\mathbb{S}}_j(t',y)\,\frac{f_j(y,t')}{f_i(x,t')}\mathbb{S}_{l(ij)}(t'(1-z)^2,y(1-z))\,. \quad (6.7)\end{aligned}$$

The notation $\bar{\mathbb{S}}$ stands for the spacelike shower, and $l(ij)$ stands for the other parton in the splitting ij (i.e., for example $l(qq)=g$, $l(gq)=\bar{q}$, etc.). Observe that the starting value of the ordering variable after the splitting is exactly t', which is consistent with our choice of the ordering variable for backward evolution, eq. (6.1). The initial condition for the final state shower is instead consistent with our conventions for timelike showers.

It is instructive to see how unitarity arises in eq. (6.7). One has

$$f_i(x,t) = \Delta_i(t,t_0)\,f_i(x,t_0) + \int_{t_0}^{t} dt'\,dz\,dy\,\Delta_i(t,t')\,F_{ij}(t',z)\delta(x-zy)\,f_j(y,t')\,. \quad (6.8)$$

Eq. (6.8), together with the definition of the Sudakov form factors[7]

$$\begin{aligned}\Delta_q(t,t_0) &= \exp\left[-\int_{t_0}^{t}\frac{dt'}{t'}\int_0^1 dz\frac{\alpha_S}{2\pi}P_{qq}(z)\right],\\ \Delta_g(t,t_0) &= \exp\left[-\int_{t_0}^{t}\frac{dt'}{t'}\int_0^1 dz\frac{\alpha_S}{2\pi}\left\{\frac{1}{2}P_{gg}(z)+n_f P_{qg}\right\}\right]\end{aligned}\tag{6.9}$$

yields the regularized Altarelli-Parisi equations

$$\frac{d}{dt}f_i(x,t) = \int_x^1 \frac{dz}{z}\,\hat{F}_{ij}(t,z)f_j\left(\frac{x}{z},t\right).\tag{6.10}$$

6.3 The SMC parton densities

Eq. (6.10) is a formulation of the Altarelli Parisi equation which is appropriate for angular ordered showers [8, 11]. It differs from the traditional formulation in the definition of the momentum variables, and also in the z dependent argument of α_S. The corresponding parton density $f^{\rm HW}$ are equivalent to the $\overline{\rm MS}$ parton densities $f^{\overline{\rm MS}}$ in collinear leading order. They however differ by dominant double log terms in the threshold region. In order to compare them, we write the expression of the Drell-Yan cross section that they yield. In the HERWIG scheme, the initial evolution angle for Drell-Yan is of order 1, and the energy of order Q (the lepton pair invariant mass), so that the initial t value is Q. Using the HW parton densities and the Born term for the partonic cross section, we get schematically, for the $N^{\rm th}$ moment of the cross section,

$$\sigma_N^{\rm DY}(Q) = \hat{\sigma}_N^{\rm DY}\left[f_N^{\rm HW}(Q)\right]^2,\tag{6.11}$$

where f_N stands for the $N^{\rm th}$ moment of the parton density. On the other hand, in the $\overline{\rm MS}$ scheme we have

$$\sigma_N^{\rm DY}(Q) = \hat{\sigma}_N^{\rm DY}f_N^{\overline{\rm MS}^2}(\mu)\exp\left\{-\frac{4C_F}{\pi}\int_0^1 dz\frac{z^N-1}{1-z}\int_{Q^2(1-z)^2}^{\mu^2}\frac{dq^2}{q^2}\alpha_S(q^2)\right\},\tag{6.12}$$

which also includes double log resummation in the large N limit [11]. Equations (6.12) and (6.11) have the same Q^2 dependence

$$Q^2\frac{\partial}{\partial Q^2}\sigma_N^{\rm DY}(Q) = \sigma_N^{\rm DY}(Q)\frac{4C_F}{\pi}\int_0^1 dz\frac{z^N-1}{1-z}\alpha_S(Q^2(1-z)^2),\tag{6.13}$$

which can be derived directly from eq. (6.12) for the $\overline{\rm MS}$ scheme, and from eqs. (6.11) and (6.10) for the HERWIG scheme. This suggests the following identification

$$f_N^{\rm HW}(t) = f_N^{\overline{\rm MS}}(\mu)\exp\left\{-\frac{2C_F}{\pi}\int_0^1 dz\frac{z^N-1}{1-z}\int_{t(1-z)^2}^{\mu^2}\frac{dq^2}{q^2}\alpha_S(q^2)\right\}.\tag{6.14}$$

[7]The factor of 1/2 in front of P_{gg} in eq. (6.9) is appropriate for the exclusive splitting functions that appear in the Sudakov form factors. It is the symmetry factor associated with the identical particles in the final state of the gluon splitting process. For the same reason only the P_{qq} function appears in Δ_q. In fact, one might replace $P_{qq}\to[P_{qq}+P_{gq}]/2$ in Δ_q, and $P_{qg}\to[P_{qg}+P_{\bar{q}g}]/2$ in Δ_g, to have more symmetric expressions.

Thus, it is possible to define the HW parton densities in such a way that they resum all Sudakov double logs arising in initial state radiation. It is not possible, however, to define them to include all double logs. For example, in the DIS process there are double logs arising from the final state jet, due to the fact that as $x \to 1$ the final state jet mass is forced to become small, thus inducing further Sudakov suppression. This leads to the well known fact that it is not possible to absorbs soft double logs universally in the parton densities.

If the final state jet is described by an angular ordered shower, the soft gluon structure of the final state is also correctly described in the DIS case. However, in order to get an accurate total rate in the threshold region, one has to correct the parton cross section with the Sudakov term associated to the final state jet [11].

When implementing an SMC+NLO program with hadrons in the initial state, one should remember that, in order to have NLO accuracy, NLO structure functions should be used. On the other hand, we have seen that for consistency a HERWIG scheme should be used for the structure functions. When discussing NLO corrections we will have to pay attention to this problem, and we will see in the following that it may be overcome in several ways.

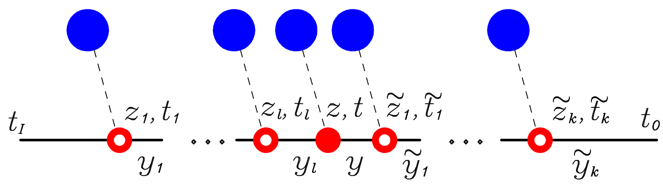
6.4 Hardest emission

In the spacelike, backward evolution shower it is obvious that the hardest emission is to be found along the spacelike line. An arbitrary number of soft emissions can occur before the hardest one. They can be implemented (as in the timelike case) as truncated, vetoed backward showers. After the hardest emission the backward shower will continue with vetoed emissions.

Having singled out the hardest emission, one has an alternative formulation of the angular ordered shower, in which the hardest emission is generated first. It would be generated with the HERWIG parton densities, and a transverse momentum Sudakov form factor. Thus, it would include some NLO corrections, more precisely the threshold corrections arising from initial state radiation. As discussed above, it would not include other threshold corrections, like those arising in DIS from final state jets. One can now correct the hardest emission, using the exact matrix elements and NLO structure functions. This will guarantee NLO accuracy for IR and collinear safe quantities. On the other hand, the rest of the initial state shower can be implemented using the LL HERWIG structure functions. This does not affect IR safe quantities, and on the other hand guarantees consistency in the treatment of soft radiation from the initial state line.

The equation for the angular ordered initial state shower can be formally solved by iteration. The hardest emission can be singled out as in eq. (4.1), yielding

$$\bar{\mathbb{S}}(t_I, x) = \Pi(t_I, t_0, x)\langle\mathbb{I}| + \sum_{l,k=0}^{\infty}$$



. (6.15)

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We will also use the notation $z_i/t_i/y_i$ for $i=1,l+k+1$ to denote collectively the $z_i/t_i/y_i$, $z/t/y$ and $\tilde{z}_i/\tilde{t}_i/\tilde{y}_i$ sequences. The graphic expression in eq. (6.15) has the following meaning.

- Thick lines are Sudakov form factors; more specifically, they are given by

$$\Pi(t_I,t_1,x),\ \Pi(t_1,t_2,y_1),\ldots,\Pi(t_{k+l+1},t_0,y_{k+l+1})\,. \tag{6.16}$$

- Each y value equals the y value to its left divided by z. Thus $y_1=x/z_1$, $y_2=y_1/z_2$, etc.

- The solid blobs are $\mathbb{S}(y_i^2(1-z_i)^2t_i)$.

- The hollow blobs stand for

$$F(z_i,t_i)\,\frac{f(y_i,t_i)}{f(z_iy_i,t_i)}\,\theta(p_{\rm T}-\sqrt{t_i}y_iz_i(1-z_i))\,. \tag{6.17}$$

- The solid vertex blob (i.e. the largest p_T vertex) is given by the same expression as the hollow blob without the theta function.

- All intermediate z_i, t_i are integrated.

As in the timelike case, eq. (6.15) does not have a simple Monte Carlo implementation, since no-branching factors do not match the branching factors. We can however introduce modified form factors, so that they match the splitting probability, and collect the leftover. More precisely, using the expression of eq. (6.6) for Π, we rewrite it as

$$\Pi(t,t',x)=\Pi_V(t,t',x)\times\exp\left\{-\int_{t'}^{t}dt''\int\frac{dz}{z}\,F(z,t'')\frac{f(x/z,t'')}{f(x,t'')}\theta(\sqrt{t''}x(1-z)-p_{\rm T})\right\}, \tag{6.18}$$

with

$$\Pi_V(t,t',x)=\exp\left\{-\int_{t'}^{t}dt''\int\frac{dz}{z}\,F(z,t'')\frac{f(x/z,t'')}{f(x,t'')}\theta(p_{\rm T}-\sqrt{t''}x(1-z))\right\}. \tag{6.19}$$

Now Π_V matches exactly the emission vertex. It can be easily implemented by generating the next evolution scale with the usual form factor, supplemented by a $p_{\rm T}$ veto procedure. The remaining factors should be collected together, yielding

$$\Pi_{\rm rem}=\exp\left[-\sum_{i=0}^{l+k+1}\int_{t_{i+1}}^{t_i}dt\,\frac{dz}{z}\,F(z,t)\frac{f(y_i/z,t)}{f(y_i,t)}\theta(\sqrt{t}y_i(1-z)-p_{\rm T})\right]. \tag{6.20}$$

The $\Pi_{\rm rem}$ term is easily shown to be equivalent to

$$\Pi_{\rm rem}\approx\Pi_R(t_I,t_0,x)=\exp\left[-\int_{t_0}^{t_I}dt\,\frac{dz}{z}\,F(z,t)\frac{f(x/z,t)}{f(x,t)}\theta(\sqrt{t}x(1-z)-p_{\rm T})\right]. \tag{6.21}$$

In fact, the theta function in eq. (6.20) prevents the appearance of any large (soft or collinear) log when $t \approx p_{\mathrm{T}}$. Now, if there is an i for which z_i is not near one, for all $j \geq i$ we have $t_i \lesssim p_{\mathrm{T}}$. In fact, at the i^{th} vertex we must have

$$t_i y_i z_i (1 - z_i) \leq p_{\mathrm{T}} \Longrightarrow t_i \lesssim p_{\mathrm{T}}\,. \tag{6.22}$$

But for all t_j with $j > i$ we have $t_j < t_i$ because of angular ordering, and so also $t_j \lesssim p_{\mathrm{T}}$. It follows that in eq. (6.20), for all terms in the sum that are not suppressed by the θ function all previous z_i values are near 1, and so the corresponding y_i value is near x. All terms in the sum have then the same integrands, and the integration ranges can be joined together to yield eq. (6.21).

6.5 NLO Correction

The SMC+NLO correction to the hardest emission is obtained as in the timelike case, and in fact it has the same expression given in eqs. (5.7) and (5.8), provided one includes in the definition of B, V, C and R the appropriate structure function factor. By doing so, in the collinear regions the R/B expression entering in the Sudakov form factor (5.7) matches the expression in eq. (6.21).

As in the case of the timelike shower, one can use p_{T} as the argument of one power of α_{S} in the expression for R appearing in the square bracket of eq. (5.10). In the same instances one can also replace the NLO parton densities with the HERWIG ones, at the price of a NNLO unknown effect, and with the advantage of a treatment of the all order double log region which is consistent with the original SMC. Thus, it is possible to perform the matching in a fully consistent way. Only practical studies of real implementations will tell whether all these subtleties are of any importance at all.

7. Comparison with other methods

I will now discuss comparisons and relations with few popular methods used to perform some sort of NLO improvement of the SMC.

7.1 Matrix element corrections

With this method one corrects the large angle emissions. Thus, large angle gluon emission become accurate at the NLO level, but the full integral of emissions at any angle has still only LO accuracy. A discussion of this method appropriate to SMC with interfering gluons has been introduced in ref. [9]. One generates the shower as usual, and applies a correction factor corresponding to the exact tree level expression divided by the SMC approximate one only if the emission is the hardest that has taken place so far. In order for this method to work, one has to make sure that the SMC fills all the phase space for gluon emission, in order not to miss any region.

The method proposed in the present work can achieve the same purpose, just by using B instead of $\bar{B}$ in eq. (5.10) so that one just needs to know the matrix elements for real emission. It has the advantage that one needs not to worry about filling the phase space for gluon emission.

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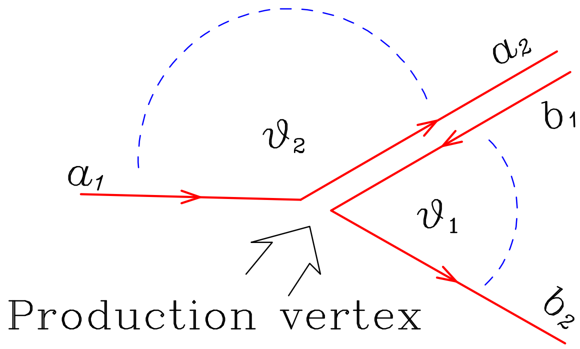

Figure 6: Radiation from the $q\bar{q}g$ system in the planar approximation.

7.2 Tree level initial state with radiation

In the SMC generation of $e^+e^- \to hadrons$ events one usually starts with a $q\bar{q}$ event as the hard event. When three jet events are studied, one can start with a $q\bar{q}g$ event, requiring a cut on some final state variable (like thrust) in order to stay away from the two-jet region. In SMC including gluon interference, one has then to set appropriate initial showering angle for each final state line. This poses no problems when the initial state lines form large angles. It may cause problems if some partons form relatively small angles. To illustrate the problem, we consider the $e^+e^- \to q\bar{q}g$ example. We draw the colour lines for this process in the planar approximation, as shown in figure 6.

The gluon has a colour and anticolour line associated with it. Thus its emission angle can be controlled either by θ_1 or by θ_2. On the other hand, in the angular ordered formalism one generates a single shower starting from a given parton, with a given initial angle. The HERWIG program chooses the initial condition for the evolution of the gluon jet by picking θ_1 or θ_2 with equal probability. In both cases the radiation probability of the gluon line is given by the gluon coupling C_A. This works properly for the next soft emission in the shower. In fact, let us examine the soft emission pattern of the $q\bar{q}g$ system in the large N limit. It is given by the incoherent sum of the emissions from the two antennas a_1, a_2 and b_1, b_2. The b antenna only radiates inside the small angle (i.e. θ_1) cone in the qg direction. The a antenna fills the whole solid angle. The colour charge of the gluon's colour lines a_2 and b_1 is C_F. Near the gluon, for emission angles below θ_1 both antennas give a collinear enhanced contribution of the same size, and their sum correspond to replacing $C_F \to 2C_F \approx C_A$. In HERWIG, near the gluon (i.e. for angles in the θ_1 cone) one has in fact radiation with charge C_A. Near the gluon, but for emission angles larger than θ_1, only 50% of the events radiate with C_A intensity. The net effect is radiation of the qg system with a C_F intensity in the large angle region. This is what we expect: the a antenna radiates with intensity C_F, and the b antenna does not radiate for angles larger than θ_1. This clever scheme, however, does not work for further emissions. In fact, lack of soft radiation around the qg system should be suppressed by a Sudakov form factor. In the HERWIG scheme we instead have just a 50% suppression. In fact, in 50% of the events the qg system does not radiate for angles greater than θ_1. This means that for pseudorapidity below the θ_1 angular region the qg system would be scarcely populated by

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soft radiation. It is not clear whether this fact may cause observable problems. At the end of the shower the coloured parton associated with the gluon colour line will have a colour partner originating from soft radiation from the a antenna. According to the argument given above this partner may not be close enough in rapidity to form a low mass colour cluster.

The method proposed in this paper suggests instead to assign an initial showering angle of θ_1 to both the quark and the gluon. One then adds a truncated shower from θ_2 to θ_1 associated with the qg system. It is clear that this procedure gives the right amount of soft radiation colour connected to the outgoing gluon line.

7.3 The CKKW formalism

The CKKW formalism [12] is a method to merge perturbative angular ordered showers with exact matrix elements. The method requires a cutoff on the cluster mass, which is used to separate the application of exact and approximate matrix elements.

There are several analogies among concepts that have arisen in the present work and in the CKKW paper. In particular, the need to introduce vetoed showers arises also there.

Here I would like to focus on one concept that has no parallel in the CKKW work: the concept of a truncated shower. In fact, CKKW avoid the need of a truncated shower using a trick[8] that I will illustrate here referring to the e^+e^- example of figure 6.

In the CKKW method one limits the gluon radiation to θ_1, but lets the quark radiate at all angles. In the truncated shower scheme one requires that both the gluon and the quark radiate starting from the angle θ_1, and then adds a truncated shower of the qg system for larger angles. Since the colour charge of the qg system is equal to the charge of the quark, the radiation of the truncated shower combines with the radiation of the quark to yield quark radiation in the whole solid angle, which is the CKKW method. This also works for gluon splitting: one lets the harder gluon radiate at all angles. The $g \to q\bar{q}$ has to be treated differently. Here one should let both quark and antiquark radiate at large angle, so that one reconstructs the C_A charge of the truncated shower by summing up two C_F's.

As far as particle flow and multiplicities, the CKKW method is equivalent to the truncated shower method. It is however different in the colour connection that it assigns to final state partons. The colour connection of a soft emission in the truncated shower is the same as if the emission came from the parent parton. This is what it should be, since it is coherent radiation. In the CKKW method colour connections are not respected, and this may cause problems with large mass colour connected parton clusters.

8. Conclusions

In the present paper I presented a method for the inclusion of NLO corrections in Monte Carlo programs with angular ordering, that avoid the problem of negative weighted event. The method may be applied also for the less ambitious goal of correcting the large angle

[8] See Sec. 3.2 of [12].

emissions in a SMC. Also in this contexts it has some advantages. The implementation of this method requires minor modifications to the showering scheme in SMC programs. One modification is the inclusion of p_T vetoing in the shower. This is common practice in SMC, and should not present particular problems. Truncated showers instead have never appeared in SMC implementations. They play a central role in the present work, since it does not seem possible to implement the soft radiation of a collinear bunch of partons without truncated showers. It is therefore desirable that in the context of rewriting or developing new Monte Carlo programs will provide facilities to implement truncated showers. Besides opening the possibility of SMC+NLO implementations for several processes, this would also allow to improve matrix element corrections schemes with a limited effort.

The method suggested here also hints at the possibility of SMC+NNLO implementations. One should isolate the two hardest emission in a SMC event, consider again p_{T} vetoed showers at the smaller p_{T} of the two hardest emissions, and add truncated shower associated with two or three partons arising from the hardest splittings. Although it may be premature to discuss this problem in details now, it may be convenient to consider the possibility of implementing truncated shower from more than two lines when building SMC programs.

The method discussed in the present paper has been developed in the context of angular ordered showers, in particular as implemented in the HERWIG SMC. The aim of this work was to provide an approach along the lines of ref. [1]–[3], improved over some substantial aspects. Different approaches to the problem of NLO improvement of SMC programs have been presented in refs. [13]–[16].

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References

[1] S. Frixione and B.R. Webber, *Matching NLO QCD computations and parton shower simulations*, *J. High Energy Phys.* **06** (2002) 029 [hep-ph/0204244].

[2] S. Frixione, P. Nason and B.R. Webber, *Matching NLO QCD and parton showers in heavy flavour production*, *J. High Energy Phys.* **08** (2003) 007 [hep-ph/0305252].

[3] S. Frixione and B.R. Webber, *The mc@nlo 2.0 event generator*, hep-ph/0307146.

[4] S. Gieseke, P. Stephens and B. Webber, *New formalism for QCD parton showers*, *J. High Energy Phys.* **12** (2003) 045 [hep-ph/0310083].

[5] S. Gieseke, A. Ribon, M.H. Seymour, P. Stephens and B. Webber, *Herwig++ 1.0: an event generator for e^+e^- annihilation*, *J. High Energy Phys.* **02** (2004) 005 [hep-ph/0311208].

[6] G. Marchesini and B.R. Webber, *Simulation of QCD jets including soft gluon interference*, *Nucl. Phys.* **B 238** (1984) 1.

[7] T. Sjostrand, *A model for initial state parton showers*, *Phys. Lett.* **B 157** (1985) 321.

[8] G. Marchesini and B.R. Webber, *Monte Carlo simulation of general hard processes with coherent QCD radiation*, *Nucl. Phys.* **B 310** (1988) 461.

[9] M.H. Seymour, *Matrix element corrections to parton shower algorithms*, *Comput. Phys. Commun.* **90** (1995) 95 [hep-ph/9410414].

[10] M.L. Mangano, P. Nason and G. Ridolfi, *Heavy quark correlations in hadron collisions at next-to-leading order*, *Nucl. Phys.* **B 373** (1992) 295.

[11] S. Catani, B.R. Webber and G. Marchesini, *QCD coherent branching and semiinclusive processes at large x*, *Nucl. Phys.* **B 349** (1991) 635.

[12] S. Catani, F. Krauss, R. Kuhn and B.R. Webber, *QCD matrix elements + parton showers*, *J. High Energy Phys.* **11** (2001) 063 [hep-ph/0109231].

[13] J. Collins, *Monte-Carlo event generators at NLO*, *Phys. Rev.* **D 65** (2002) 094016 [hep-ph/0110113].

[14] J.C. Collins, *Subtraction method for NLO corrections in Monte Carlo event generators for leptoproduction*, *J. High Energy Phys.* **05** (2000) 004 [hep-ph/0001040].

[15] M. Kramer and D.E. Soper, *Next-to-leading order QCD calculations with parton showers, I. Collinear singularities*, *Phys. Rev.* **D 69** (2004) 054019 [hep-ph/0306222].

[16] D.E. Soper, *Next-to-leading order QCD calculations with parton showers, II. Soft singularities*, *Phys. Rev.* **D 69** (2004) 054020 [hep-ph/0306268].